A singular Sturm-Liouville problem treated by non-standard analysis.

by

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1. Introduction.

In a recent paper, S. Albeverio, J.E. Fenstad, and J.R. Høegh-Krohn [2] prove that the classical theorems on the oscillation of eigenfunctions of the Sturm-Liouville problem for second-order differential equations remain true when the zero-order coefficient is given by a measure instead of by a (continuous) function. This generalization is motivated by applications to physics.

Their proof uses the classical Sturm-Liouville theory (smooth coefficients), a certain amount of perturbation theory for operators in Hilbert space, and the "Transfer principle" (or "Elementary extension Principle") of non-standard analysis.

On the other hand, A.L. MacDonald, in [6], gives a new proof of the completeness of the eigenfunctions for the classical (regular) Sturm-Liouville problem. His idea is to approximate the differential equation by a difference equation in the "obvious" way, and to use the fact that for the corresponding finite-dimensional eigenvalue problem the completeness of the eigenvectors is trivial. The core of his argument is an inequality which enables him to "pass from the discrete to the continuous case" via the Transfer principle.

This approach is conceptually pleasing, and if one accepts the Transfer Principle, it is also technically much simpler than the classical proofs of completeness.

The aim of the present note is to show that the method of finite differences, as used by MacDonald in [5], can be refined to work in the more sophisticated setting of Albeverio et al. This will give new and simpler proofs of the results in [2, Section 4], and somewhat better bounds on the eigenfunctions.

The paper is organized as follows:

Our main results are stated and commented upon in Section 2. Section 3 contains a brief summary of some more or less elementary facts about difference equations. The crucial inequalities are proved in Section 4, while the passage from the discrete to the continuous case, via non-standard analysis, will be found in Section 5.

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2. The problem, and the results.

We let μ denote a finite non-negative Borel measure on [0,1], and consider the eigenvalue problem

$$-Y''(x) + \mu Y(x) = \tau Y(x), \quad 0 \leq x \leq 1 \tag{2.1}$$

$$Y(0) = Y(1) = 0 \tag{2.2}$$

where τ is a parameter.

There are several ways to give a precise meaning to (2.1). One of them is to multiply in (2.1) by $Y(x)$ and integrate over $0 \leq x \leq 1$. Using integration by parts, and (2.2), one is led to consider the quadratic form defined by

$$A\Phi = \int_0^1 (\Phi')^2 \, dm + \int_0^1 \Phi^2 \, d\mu \, . \tag{2.3}$$

(where dm denotes Lebesgue measure), on the space C_0^1 of those continuously differentiable functions Φ on [0,1] which satisfy (2.2). We will prove that A has a countable family of "generalized eigenfunctions" $\{Y_j\}$ which behave very much in the same way as do the eigenfunctions of the classical Sturm-Liouville problem; except that they will not, in general, be differentiable. A precise statement is found in Theorem 1 below.

Another reasonable interpretation of (2.1) is obtained by integrating twice over an interval $[0,x]$, and then change the order of integration. This leads to the integral equation

$$Y(x) = xY'(0) + \int_0^x (x-s)Y(s)\,d\mu - \tau \int_0^x (x-s)Y(s)\,dm. \tag{2.4}$$

It is not hard to prove by standard methods ("contraction principle") that for every τ, (2.4) has a continuous solution Y_τ on $[0,1]$, but it seems difficult by such methods to decide for which τ we have $Y_\tau(1) = 0$, and to obtain further information about these eigenfunctions. We will prove that the above-mentioned Y_j, τ_j solve this problem too.

Theorem 1. Let μ be a finite Borel measure on $[0,1]$, and define a quadratic form A on $C_0^1[0,1]$ by (2.3). Write $M = \mu[0,1]$, $B = 2(1+12M)$.

There exists a sequence $\{\tau_j\}_{j=1}^{\infty}$ of real numbers, and a sequence $\{Y_j\}_{j=1}^{\infty}$ of continuous functions on $[0,1]$, such that

a) The following inequalities hold:

$$\pi^2 j^2 \leq \tau_j \leq (\pi j + M)^2$$

and

$$|Y_j(x)| \leq B \qquad 0 \leq x \leq 1, \quad 1 \leq j < \infty$$

b) $\{Y_j\}$ is an orthonormal and complete sequence in $L^2[0,1]$.

c) If Φ is twice continuously differentiable on $[0,1]$, and $\Phi(0) = \Phi(1) = 0$, then its orthogonal expansion in terms of $\{Y_j\}$ converges uniformly to Φ.

d) If Φ is continuously differentiable on $[0,1]$, $\Phi(0) = \Phi(1) = 0$, and $\sum_{j=1}^{\infty} d_j Y_j$ is its expansion in terms of $\{Y_j\}$, then

$$A\Phi = \sum_{j=1}^{\infty} \tau_j d_j^2$$

e) The Y_j are solutions of (2.4) with $\tau = \tau_j$.

f) Y_j has exactly $j+1$ zeroes in the closed interval $[0,1]$, and between two zeroes for Y_j there is a zero for Y_{j+1}.

g) If, on some interval $I \subset [0,1]$, the restriction of μ to I is absolutely continuous with respect to Lebesgue measure, then the Y_j are continuously differentiable on I, the Y_j' are absolutely continuous, and the (τ_j, Y_j) solve (2.1) in the ordinary sense, almost everywhere on I. On such an interval

$$|Y_j'(x)| \leq 2B\,\tau_j^{\frac{1}{2}}$$

Remark 1. Albeverio et al., in [2] consider a seemingly more general problem: they define the quadratic form A by

$$A\Phi = \int_0^1 \Phi'(x)^2 P(x)\,dm + \int_0^1 \Phi(x)^2 d\mu$$

where P is a measurable non-negative function on $[0,1]$, with $1/P$ integrable. It is not hard to verify that the change of variable

$$t(x) = \int_0^x \frac{ds}{P(s)}$$

will reduce this to the case $P \equiv 1$.

Remark 2. For $M < \tau_j < \infty$ the remark following Proposition 3 implies a sharper estimate for the Y_j:

$$\|Y_j\|_\infty \leq \sqrt{2}(2 + 2M)(1 - M\tau_j^{-\frac{1}{2}})^{-1}.$$

Also, asymptotic estimates of the form

$$\tau_j - j^2\pi^2 \leq C \cdot j^{-1}$$

can be proved by adapting the method of [4, § 11.4] to the identity (4.5), and then proceeding as in Section 5.

3. The discrete boundary value problem.

A Sturm-Liouville theory for the difference equation

$$N^2\Delta^2 y(k) + (\lambda - q(k))y(k) = 0, \qquad k = 1,2,\dots,N-1 \tag{3.1}$$

with boundary conditions

$$y(0) = y(N) = 0 \tag{3.2}$$

can be developed by essentially the same methods as for the corresponding differential equation. The main difference is that since in the discrete case one works in a finite-dimensional space, the proof of the completeness of the family of eigenfunctions is much easier.

All this must have been known for nearly a century, but we have found no convenient reference for the discrete version of the theory, so in this section we give a brief summary of it, as far as necessary for our present purposes.

We use the following notations: N is a fixed positive integer ≥ 5. The "potential" $q = \{q(1),\dots,q(N-1)\}$ is given, it is supposed to be non-negative:

$$q(k) \geq 0, \qquad 1 \leq k < N, \tag{3.3}$$

and it will sometimes be convenient to define $q(k) = 0$ for $k = 0$, and for $k \geq N$. λ is a real parameter. The difference operators Δ and Δ^2 are defined by

$$\Delta y(k) = y(k+1) - y(k)$$

and, to preserve some symmetry in the formulae

$$\Delta^2 y(k) = \Delta(\Delta y(k-1)) = y(k+1) - 2y(k) + y(k-1)$$

The factor N^2 in (3.1) could of course have been absorbed into q and λ, but in Section 5 it will be slightly more convenient to have it the way we have written it.

We will use the following norms for vectors $v = \{v(1),\dots,v(N)\}$:

$$\|v\|_1 = \sum_{k=1}^{N} |v(k)|, \quad \|v\|_2 = (\sum_{k=1}^{N} |v(k)|^2)^{\frac{1}{2}}, \quad \|v\|_\infty = \max_{1\leq k\leq N} |v(k)|$$

and recall the inequalities

$$\|v\|_\infty \leq \|v\|_1 \leq N^{\frac{1}{2}}\|v\|_2 \leq N\|v\|_\infty.$$

The equations (3.1), (3.2) can of course be considered as a system of $N-1$ linear equations for the $N-1$ real unknowns $y(1),\dots,y(N-1)$. The corresponding matrix is of the form

$$A - \lambda I$$

where I is the identity matrix, and where the entries A_{ij} of A are:

$$\begin{aligned} A_{ij} &= 0 \quad \text{if} \quad |i-j| > 1 \\ A_{ij} &= N^2 \quad \text{if} \quad |i-j| = 1 \\ A_{ii} &= -2N^2 - q(i), \quad 1 \leq j < N. \end{aligned} \tag{3.4}$$

Since the matrix A is symmetric, it follows from elementary linear algebra that (3.1), (3.2) has $N-1$ pairwise orthogonal real eigenvectors $y_1,\dots,y_{N-1}$, which form a basis for $\mathbb{R}^{N-1}$ and that the corresponding eigenvalues $\lambda_1,\dots,\lambda_{N-1}$ are real.

The eigenvalues are also simple. If y_1 and y_2 are solutions of (3.1) (for one and the same value of λ), and if $y_1(0) = y_2(0) = 0$, then y_1 and y_2 are proportional

$$y_2(k) = y_1(k) \ \ y_2(1)/y_1(1), \qquad k \geq 1$$

as is seen from (3.1) by induction in k.

For later reference, we sum up all this as

Proposition 1. The problem (3.1), (3.2) has $N-1$ real, simple eigenvalues, which we denote by

$$\lambda_1 < \lambda_2 < \dots < \lambda_{N-1}$$

The corresponding eigenvectors, $y_1,\dots,y_{N-1}$, which we normalize by

$$\|y_j\|_2^2 = \sum_{k=1}^{N-1} y_j(k)^2 = N \tag{3.5}$$

are mutually orthogonal: If $i \neq j$, then

$$\langle y_i, y_j\rangle = \sum_{k=1}^{N-1} y_i(k) y_j(k) = 0 \tag{3.6}$$

and they span $\mathbb{R}^{N-1}$: any vector $\varphi = \{\varphi(k)\}_{k=1}^{N-1} \in \mathbb{R}^{N-1}$ has a unique expansion

$$\varphi(k) = N^{-1}\sum_{j=1}^{N-1} c_j y_j(k) \qquad 0 < k < N . \tag{3.7}$$

The coefficients c_j are given by

$$c_j = \langle \varphi, y_j\rangle = \sum_{k=1}^{N-1} \varphi(k) y_j(k).$$

The special case of (3.1), (3.2) where q is constant, can be solved explicitly in terms of elementary functions. We rewrite it as

$$N^2\Delta^2 z(k) + \sigma z(k) = 0 \tag{3.9}$$

$$z(0) = z(N) = 0 \tag{3.10}$$

and state, for later reference, some facts which will be useful. The proofs are straightforward and elementary, but not very illuminating, so we omit them.

When $0 < \sigma < 4N^2$, all real solutions of (3.9) can be written in the form

$$(3.11) \qquad z(k) = c\sin((k-\varkappa)\alpha/N)$$

where c and $\varkappa$ are arbitrary constants, and where $\alpha = \alpha(\sigma)$ is defined by

$$(3.12) \qquad \cos(\alpha/N) = 1 - \sigma/2N^2, \quad 0 < \alpha/N < \pi.$$

The function $\alpha(\sigma)$ defined by (3.12) satisfies the identity

$$(3.13) \qquad N^2\sin^2(\alpha/N) = \sigma(1 - \sigma/4N^2)$$

and the inequalities

$$(3.14) \qquad \alpha^2/6 < \alpha^2(1 - \frac{\alpha^2}{12N^2}) < \sigma < \alpha^2.$$

For $0 < \sigma < 3N^2$, we also have

$$(3.15) \qquad \sigma/4 < N^2\sin^2(\alpha/N) < \sigma$$

For $c = 1$, $\varkappa = 0$ in the solution (3.12), we have the inequalities

$$(3.16) \qquad N/3 \leq \sum_{k=1}^{N} \sin^2(k\alpha/N) \leq N$$

and

$$(3.17) \qquad \sigma/3 \leq N\sum_{k=1}^{N} (\Delta\sin(k\alpha/N))^2 \leq \sigma.$$

Remark. For large N, (3.16) and (3.17) can be sharpened: For any constant $c > 1$ there exists an N_c such that when $N > N_c$, the upper bounds N and σ can be replaced by $cN/2$ and $c\sigma/2$, respectively. Also, for large N, the lower bounds can be replaced by $N/2$ and $\sigma/2$, respectively, when $0 < \sigma < N^2$.

The eigenfunctions z_j, $1 \leq j < N$, for the problem (3.9), (3.10) are, (up to a normalization factor):

(3.18) $$z_j(k) = \sin(kj\pi/N) \qquad 0 \le k \le N.$$

and the corresponding eigenvalues are

(3.19) $$\sigma_j = 2N^2(1-\cos(j\pi/N)), \qquad 1 \le j < N.$$

The σ_j satisfy the following inequalities:

$$j^2\pi^2/6 < j^2\pi^2(1 - j^2\pi^2/12N^2) < \sigma_j < j^2\pi^2$$

and, (if $N \ge 4$):

(3.21) $$\sigma_1 > 9.$$

We also have explicit values for the norms of the z_j, and of their differences:

(3.22) $$\|z_j\|_2^2 = \sum_1^N \sin^2(kj\pi/N) = N/2$$

and

(3.23) $$\|\Delta z_j\|_2^2 = \sigma_j N/2.$$

Sturm's classical oscillation and separation theorems are also just as easy (or just as hard) to prove for difference equations as they are for differential equations. There is one point which should be mentioned; the notion of a zero-point for a sequence $y = \{y(k)\}$ must be made precise. We do that by linear interpolation: If, for some integer k, $y(k)y(k+1) \le 0$ and $\Delta y(k) \ne 0$, then the real number

$$\xi = k - y(k)/\Delta y(k)$$

is called a <u>zero-point</u> or a <u>node</u> for y.

<u>Note</u> that under these circumstances $0 \le -y(k)/\Delta y(k) \le 1$, and that if y solves (3.1), then $y(k)y(k+1) \le 0$ <u>implies</u> $\Delta y(k) \ne 0$ (unless $y \equiv 0$).

We will need the following facts.

Proposition 2. Let λ_j and $y_j = \{y_j(k)\}_0^N$ be the j-th eigenvalue and eigenvector for (3.1), (3.2). Then:

a) y_j has exactly $j+1$ nodes in the closed interval $[0,N]$.

b) Between two nodes for y_j there is a node for y_{j+1}.

c) If ξ_1 and $\xi_2 > \xi_1$ are two consecutive nodes for y_j, then

(3.25) $$N\pi(6\lambda_j)^{-\frac{1}{2}} \leq \xi_2 - \xi_1 \leq N\pi(\lambda_j - \|q\|_\infty)^{-\frac{1}{2}}$$

(the right-hand inequality only if $\lambda_j > \|q\|_\infty$.)

d) the eigenvalues λ_j satisfy the following inequalities, where σ_j is defined by (3.19), and $m = \|q\|_1/N$:

(3.26) $$j^2\pi^2/6 < \sigma_j \leq \lambda_j \leq \sigma_j + \|q\|_\infty < j^2\pi^2 + \|q\|_\infty .$$

(3.27) $$\sum_{j=1}^{N-1} \lambda_j^{-1} < 1$$

and, for $1 \leq j \leq N$:

(3.28) $$\lambda_j \leq (j\pi + m)^2.$$

Proof. a) and b) are just Satz 1 and Satz 4 in Chapter II, § 1 of Gantmacher and Krein's book [3].

To prove c) note that the proof of Satz 2, in the same section of [3] can be modified so as to prove the following:
If y is a solution of (3.1), and if $\tilde{y}$ is a solution of the same equation with λ changed to $\tilde{\lambda} > \lambda$ or q changed to $\tilde{q} \leq q$, then between two nodes for y there is at least one node for $\tilde{y}$. Then c) follows by comparing solutions of (3.1) with solutions of the constant coefficient equation (3.9), first with $\sigma = \lambda$ and then with $\sigma = \lambda - \|q\|_\infty$.

To prove d) we use § 9 of Ch. II in [3]. There it is proved (equ. (132)) that each λ_j is a differentiable function of the $q(k)$, with

$$\frac{\partial \lambda_j}{\partial q(k)} = y_j(k)^2/\|y_j\|_2^2. \tag{3.29}$$

In particular, λ_j is non-decreasing as a function of q, and hence (3.26) follows by comparing with the constant-coefficient case, where the eigenvalues are given by (3.19), and satisfy (3.20).

Now (3.27) follows directly from the left-hand part of (3.26).

Finally, to prove (3.28), let $\lambda_{j,\varepsilon}$ and $y_{j,\varepsilon}$ be the j-th eigenvalue and eigenvector for the problem.

$$N^2\Delta^2 y(k) + (\lambda - \varepsilon q(k))y(k) = 0$$
$$y(0) = y(N) = 0$$

where $0 \leq \varepsilon \leq 1$. For $\varepsilon = 0$, this is just (3.9), (3.10); the solution of which is given in detail above. For $\varepsilon = 1$, we have (3.1), (3.2). From (3.29) we deduce

$$\frac{d\lambda_{j,\varepsilon}}{d\varepsilon} = \sum_{k=0}^{N} q(k) y_{j,\varepsilon}(k)^2/\|y_{j,\varepsilon}\|_2^2.$$

In the next section we will prove that

$$\Sigma q(k) y_{j,\varepsilon}(k)^2 \leq \lambda_{j,\varepsilon}^{\frac{1}{2}}\, m\|y_{j,\varepsilon}\|_2^2$$

(see Lemma 4). Then (3.28) follows by integration with respect t to ε between 0 and 1.

4. Bounds for the eigenfunctions.

In this section we will prove the crucial inequalities for the eigenvectors.

Proposition 3. Let $\lambda_1 < \lambda_2 < \dots < \lambda_{N-1}$ be the eigenvalues for the problem (3.1), (3.2), and let $y_1, y_2, \dots, y_{N-1}$ be the corresponding normalized eigenvectors. Define $m = \|q\|_1/N$, and $b = 2(1+12m)$.

Then, if $\lambda_j \leq 3N^2$, in particular if $j \leq N/2$, $\|q\|_\infty \leq N$

(4.1) $$\|y_j\|_\infty \leq b$$

(4.2) $$\|\Delta y_j\|_2 \leq (\lambda_j/N)^{\frac{1}{2}}$$

(4.3) $$N\|\Delta y_j\|_\infty \leq 2\lambda_j^{\frac{1}{2}} b \ .$$

If $3N^2 < \lambda_j$, an inequality of the form (4.1) still holds, with m and b replaced by

(4.4) $$m' = \|q\|_\infty - m\,, \quad b' = 2(1+12m').$$

Remark. When N is large, and $m^2 < \lambda < N$, a sharper version of (4.1) is true. For any constant $c > 1$ there exists a N_c such that when $N > N_c$, $(cm)^2 < \lambda_j < N$:

$$\|y_j\|_\infty \leq \sqrt{2}(1+2cm)(1-cm\lambda_j^{-\frac{1}{2}})^{-1} .$$

See Remark 1 after Lemma 2 below.

For the proof of Proposition 3 we will use the following discrete version of a well-known identity from the theory of ordinary differential equations (see for instance Ince, [4,Ch.X]).

Lemma 1. Let y and z be solutions of the difference equations

$$N^2\Delta^2 y(k) = (\lambda - q(k))y(k) = 0 , \qquad k \geq 0$$

and

$$N^2\Delta^2 z(k) + \lambda z(k) = 0 , \qquad k \geq 0$$

respectively, and suppose that

$$y(0) = z(0) = 0 .$$

Then

$$y(k)z(1) = y(1)z(k) + N^{-2}\sum_{i=1}^{k-1} q(i)y(i)z(k-i). \tag{4.5}$$

Proof. Direct verification.

The difficult part of the proof of (4.1) is to obtain a sufficiently strong estimate for the sum in the right-hand side of (4.5). We leave that part aside for a moment, and present the remaining part of the proof of (4.1), along the lines of MacDonald [5].

Lemma 2. Let y, z, q, and λ be as in the previous lemma, and suppose that $0 < \lambda \leq 3N^2$. Let P be some real number such that

$$\left|\sum_{i=1}^{k} q(i)y(i)z(k-i)\right| \leq P\|y\|_2\|z\|_\infty N^{\frac{1}{2}} . \tag{4.6}$$

Then

$$\|y\|_\infty \leq 2(1 + 3P\lambda^{-\frac{1}{2}})\|y\|_2 N^{-\frac{1}{2}} . \tag{4.7}$$

Proof. Since (4.5) is homogeneous in z, we may take $z(k) = \sin(\alpha k/N)$, with $\alpha = \alpha(\lambda)$ defined by (3.12). Then (3.15) implies

$$Nz(1) \geq \tfrac{1}{2}\lambda^{\frac{1}{2}} . \tag{4.8}$$

We rearrange the terms in (4.5) and then introduce (4.6) and (4.8):

$$|y(1)/z(1)|\,|z(k)| \leq |y(k)| + |Nz(1)|^{-1} P\|y\|_2 N^{-\frac{1}{2}}$$
$$\leq |y(k)| + 2\lambda^{-\frac{1}{2}} P\|y\|_2 N^{-\frac{1}{2}}.$$

The triangle inequality in $l^2(\mathbb{R}^N)$ then implies (since $\|1\|_2^2 = N$) that

$$|y(1)/z(1)|\,\|z\|_2 \leq (1 + 2\lambda^{-\frac{1}{2}} P)\,\|y\|_2 .$$

From (3.16) we find that $\|z\|_2^2 \geq N/3 > N/4$, and thus

$$|y(1)/z(1)| \leq 2(1 + 2P\lambda^{-\frac{1}{2}})\|y\|_2 N^{-\frac{1}{2}}. \tag{4.9}$$

Finally, use (4.5) once more:

$$|y(k)| \leq |y(1)/z(1)| + |Nz(1)|^{-1}\, P\|y\|_2 N^{-\frac{1}{2}}$$
$$\leq 2(1 + 3P\lambda^{-\frac{1}{2}})\|y\|_2 N^{-\frac{1}{2}}$$

and the lemma is proved.

<u>Remark 1</u>. If N is large, (3.16) can be sharpened to $\|z\|_2^2 \geq N/2$, and if $\lambda \leq N$, (3.13) implies

$$|Nz(1)|^{-1} \leq \lambda^{-\frac{1}{2}}(1 - 1/4N)^{-\frac{1}{2}}.$$

The proof of Lemma 2 then shows that, with $c = (1 - 1/4N)^{-\frac{1}{2}}$

$$|y(1)/z(1)| \leq 2^{\frac{1}{2}}(1 + cP\lambda^{-\frac{1}{2}})\|y\|_2 N^{-\frac{1}{2}} .$$

Using (4.5) in a slightly different way, we then find

$$|y(k)| \leq 2^{\frac{1}{2}}(1 + cP\lambda^{-\frac{1}{2}}) + c\lambda^{-\frac{1}{2}}\|y\|_\infty \|q\|_1/N$$

or, if $\lambda > (c\,m)^2$:

$$\|y\|_\infty \leq 2^{\frac{1}{2}}(1 + CP\lambda^{-\frac{1}{2}})(1 - cm\lambda^{-\frac{1}{2}})^{-1}\|y\|_2 N^{-\frac{1}{2}}$$

where $c \leq (1 + 1/4N)$.

Remark 2. A natural way to obtain an inequality of the form (4.6), would be to use the Cauchy-Schwarz inequality:

$$\left|\sum_{i=1}^{k} q(i)y(i)z(k-i)\right| \leq \|q\|_2\|y\|_2\|z\|_\infty .$$

Thus, we might use $P = \|q\|_2 N^{-\frac{1}{2}}$ or, somewhat weaker: $P = \|q\|_\infty$. This latter value is the one which MacDonald uses in [6]. Unfortunately this value, although perfectly good for the case treated in [6], is too weak for our purposes.

The following trick has been adapted from S. Agmon's book [1].

Lemma 3. Let $y = \{y(0),\ldots,y(N)\}$, z, and q be vectors in $\mathbb{R}^{N+1}$, and suppose that either

(4.10) $\quad y(0) = z(N) = 0, \quad$ or $\quad y(0) = y(N) = 0.$

Then, for every real $\varepsilon > 0$:

$$\left|\sum_{i=1}^{N} q(i)y(i)^2\right| \leq \|q\|_1(\varepsilon\|\Delta y\|_2^2 + \varepsilon^{-1}\|y\|_2^2) \tag{4.11}$$

and

$$\left|\sum_{i=1}^{N} q(i)y(i)z(i)\right| \leq \|q\|_1(\varepsilon\|\Delta y\|_2 + \varepsilon^{-1}\|y\|_2)(\varepsilon\|\Delta z\|_2 + \varepsilon^{-1}\|z\|_2) \tag{4.12}$$

Proof. Define, for $0 \leq k \leq N$:

$$Q(k) = \sum_{i=o}^{k} q(i)$$

and note that $\Delta Q(k-1) = q(k)$ when $1 \leq k \leq N$.

If $v = \{v(0),\ldots,v(N)\} \in \mathbb{R}^{N+1}$, with $v(0) = v(N) = 0$, the "summation by parts" formula gives

$$\left|\sum_{i=1}^{N-1} q(i)v(i)\right| = \left|\sum_{i=1}^{N-1} \Delta Q(i-1)v(i)\right| =$$

$$= \left|\Sigma\, Q(i)\Delta v(i)\right| \leq \|Q\|_\infty\|\Delta v\|_1 .$$

Now, in view of (4.12), $v(i) = y(i)z(i)$ will satisfy the boundary condition, and we find, using the Cauchy-Schwarz inequality:

$$\|\Delta(y(i)z(i))\|_1 = \sum_{i=1}^{N-1} |\Delta y(i)z(i) + y(i)\Delta z(i)|$$

$$\leq \|\Delta y\|_2 \|z\|_2 + \|y\|_2 \|\Delta z\|_2 .$$

Finally, note that $\|Q\|_\infty \leq \|q\|_1$, and use the following two simple inequalities between non-negative reals a, b, c, d, and a positive ε:

$$2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$$

$$ab + cd \leq (\varepsilon a + \varepsilon^{-1} c)(\varepsilon d + \varepsilon^{-1} b) .$$

The lemma follows.

Lemma 4. Let y be an eigenvector for (3.1), (3.2), and λ the corresponding eigenvalue. Then

$$N\|\Delta y\|_2 \leq \lambda^{\frac{1}{2}} \|y\|_2 \tag{4.13}$$

and

$$\sum_{0}^{N} q(k)y(k)^2 \leq \lambda^{\frac{1}{2}} m \, \|y\|_2^2 \tag{4.14}$$

with $m = \|q\|_1/N$.

Proof. Multiply in (3.1) by $y(k)$, take the sum over k, $0 < k < N$, and use summation by parts:

$$N^2 \sum_{k=0}^{N-1} (\Delta y(k))^2 = \lambda \sum_{1}^{N-1} y(k)^2 - \sum_{1}^{N-1} q(k)y(k)^2 .$$

Since $q \geq 0$, this proves (4.13). To prove (4.14) use (4.11):

$$\sum q(k)y(k)^2 \leq \|q\|_1 (\varepsilon \|\Delta y\|_2^2 + \varepsilon^{-1} \|y\|_2^2) \leq \|q\|_1 (\varepsilon\lambda/N^2 + \varepsilon^{-1}) \|y\|_2^2$$

and take $\epsilon = N\lambda^{-\frac{1}{2}}$.

Lemma 5. Let y be an eigenvector for (3.1), (3.2), and let $z(k) = \sin(k\alpha(\lambda)/N)$ with $\alpha = \alpha(\lambda)$ defined by (3.12). Then

$$(4.15) \qquad |\Sigma\, q(i)y(i)z(k-i)| \leq 4\lambda^{\frac{1}{2}} m\, \|y\|_2 N^{\frac{1}{2}}.$$

(That is: We may use $P = 4\, m\, \lambda^{\frac{1}{2}}$ in Lemma 2)

Proof. Use (4.12) with $z(i)$ replaced by $z(k-i)$: N by k, $1 \leq k \leq N$. Then for $\epsilon > 0$:

$$S \equiv |\Sigma\, q(i)y(i)z(k-i)| \leq \|q\|_1(\epsilon\|\Delta y\|_2 + \epsilon^{-1}\|y\|_2)(\epsilon\|\Delta z\|_2 + \epsilon^{-1}\|z\|_2).$$

Recall from (3.16), (3.17) and Lemma 4 that

$$\|z\|_2 \leq N^{\frac{1}{2}}, \quad \|\Delta z\|_2 \leq (\lambda/N)^{\frac{1}{2}}, \quad \|\Delta y\|_2 \leq \lambda^{\frac{1}{2}}\, \|y\|_2/N.$$

This gives

$$S \leq \|q\|_1(\epsilon\lambda^{\frac{1}{2}}/N + \epsilon^{-1})\|y\|_2(\epsilon(\lambda/N)^{\frac{1}{2}} + \epsilon^{-1} N^{\frac{1}{2}}).$$

For $\epsilon = N^{\frac{1}{2}}\, \lambda^{-1/4}$, the lemma follows.

Remark. For large N, the factor 4 in (4.15) can be replaced by 2, since (3.12) and (3.13) can be improved then.

End of proof of Proposition 3. We keep the condition $0 < \lambda \leq 3N^2$ from Lemma 3 for some time yet. It follows from (3.26) that $\lambda_j \leq 3N^2$ whenever $j \leq N/2$. and $\|q\|_\infty \leq N^2/2$.

The inequality (4.2) was proved in Lemma 4, and (4.1) follows by combining Lemma 2 and Lemma 5.

To prove (4.3), start by taking differences in (4.5):

$$(4.16) \quad \Delta y(k) = (y(1)/z(1))\Delta z(k) + (Nz(1))^{-1}N^{-1}\sum_{i=1}^{k-1} q(i)y(i)\Delta z(k-i).$$

Recall the following inequalities for the various terms in (4.16):

$$|y(1)/z(1)| \leq 2(1+8\|q\|_1/N)\|y\|_2 N^{-\frac{1}{2}}.$$

follows from (4.9) and (4.15). Familiar trigonometric identities and (3.13) imply

$$|\Delta z(k)| = |\Delta \sin(k\alpha/N)| = 2|\sin(\alpha/2N)||\cos((2k+1)\alpha/2N)|$$
$$\leq 2|\sin(\alpha/2N)| = (2(1-\cos(\alpha/N)))^{\frac{1}{2}} = \lambda^{\frac{1}{2}} N^{-1}.$$

From (3.15), or (4.8) we have

$$|Nz(1)| > \lambda^{\frac{1}{2}}/2.$$

For

$$T \equiv \left|\sum_{i=1}^{k} q(i)y(i)\Delta z(k-i)\right|$$

we use (4.12): For any $\epsilon > 0$ we have:

$$T \leq \|q\|_1(\epsilon\|\Delta y\|_2 + \epsilon^{-1}\|y\|_2)(\epsilon\|\Delta^2 z\|_2 + \epsilon^{-1}\|\Delta z\|_2).$$

An inequality for $\|\Delta^2 z\|_2$ is obtained from (3.9) and (3.17):

$$N^2\|\Delta^2 z\|_2 = \lambda\|z\|_2 \leq \lambda N^{\frac{1}{2}}.$$

For the other terms in the expression for T use (3.16), (3.17), and Lemma 4: This gives

$$T \leq \|q\|_1\|y\|_2(\epsilon\lambda^{\frac{1}{2}}/N + \epsilon^{-1})(\epsilon\lambda N^{-3/2} + \epsilon^{-1}(\lambda/N)^{\frac{1}{2}})$$

and, with $\epsilon = N^{\frac{1}{2}}\lambda^{-1/4}$:

$$T \leq 4\|q\|_1\|y\|_2\lambda N^{-3/2}.$$

When all this is substituted into (4.16), the result is, when $\|y\|_2 = N^{\frac{1}{2}}$.

$$|\Delta y(k)| \leq 2(1+12\|q\|_1/N)\lambda^{\frac{1}{2}}N^{-1}$$

and (4.3) is proved.

To treat the case $\lambda_j > 3N^2$, we associate with $y = \{y(k)\}$ a new vector y^-, defined by

$$y^-(k) = (-1)^k y(k).$$

Then a simple computation shows that y solves (3.1) if and only if y^- solves

$$N^2\Delta^2 y^-(k) + (4N^2 - \lambda + q(k))y^-(k) = 0$$

which can be written in the form

$$N^2\Delta^2 y^-(k) + (\lambda^- - q^-(k))y^-(k) = 0 \tag{4.18}$$

with

$$\lambda^- = 4N^2 - \lambda - \|q\|_\infty, \quad g^-(h) = \|q\|_\infty - q(k).$$

Then (4.18) is of the same form as (3.1), with $0 \leq q^-(k) \leq \|q\|_\infty$, and

$$\|q^-\|_1/N = \|q\|_\infty - \|q\|_1/N .$$

It follows that (4.1) holds with the modified value (4.4) for b and since $\|y^-\|_\infty = \|y\|_\infty$, the proof of Proposition 3 is complete.

5. The non-standard argument. Proof of Theorem 1.

Information about non-standard analysis can be found for instance in Keisler's book [5].

We let ${}^*\mathbb{R}$ be a non-standard extension of the reals, choose a hyperfinite positive integer N, and use the following "obvious" correspondence between functions on $[0,1]$ and vectors in ${}^*\mathbb{R}^{N+1}$: To a real function Φ on $[0,1]$ we associate the vector φ defined by

$$(5.1) \qquad \varphi(k) = {}^*\Phi(k/N), \quad 0 \leq k \leq N$$

where ${}^*\Phi$ is the *-extension of Φ.

Then the following is true (see [5]):

If Φ is continuous, then

$$(5.2) \qquad \Phi(k) \sim \Phi(l) \quad \text{whenever} \quad (k-l)/N \sim 0$$

(the symbol $\sim$ denotes "infinitesimally near").

Conversely, if $\varphi \in {}^*\mathbb{R}^{N+1}$ satisfies (5.2) and if $|\varphi(k)| < \infty$ for every k, then the standard real function

$$(5.3) \qquad \Phi(x) = \mathrm{st}(\varphi(k)) \quad \text{when} \quad x = \mathrm{st}(k/N)$$

is well-defined and continuous on $[0,1]$ ($\mathrm{st}(\cdot)$ denotes "standard part").

If Φ is continuous (or at least piecewise continuous) on $[0,1]$, then

$$(5.4) \qquad \int_0^1 \Phi\,dm = \mathrm{st}\Big(N^{-1} \sum_{k=0}^{N} \varphi(k)\Big)$$

(Recall that dm denotes Lebesgue measure).

It follows from (5.4) that Φ is continuously differentiable on $[0,1]$ if and only if both φ and $N\Delta\varphi$ satisfy (5.2), and in that case

$$(5.5) \qquad \Phi'(x) = st(N\Delta\varphi(k)) \quad \text{when} \quad x = st(k/N).$$

For reference, we also note the corresponding expression for second derivatives:

$$(5.6) \qquad \Phi''(x) = st(N^2\Delta^2\varphi(k)) \quad , \quad x = st(k/N)$$

provided that $N^2\Delta^2\varphi$ also satisfies (5.2)

The representation (5.4) of integrals by Riemann sums may of course be generalized to Stieltjes integrals: If μ is some finite (Borel) measure on $[0,1]$ and ${}^*\mu$ its non-standard extension (defined via the extension of the cumulative distribution of μ) then

$$(5.7) \qquad \int_0^1 \Phi d\mu = st\Big(\sum_{k=0}^{N} {}^*\Phi(k/N)\,{}^*\mu[\tfrac{k}{N}, \tfrac{k+1}{N}\rangle\Big).$$

But the numbers ${}^*\mu[\frac{k}{N}, \frac{k+1}{N}\rangle$ may be too large for our purposes, so we need a modified version of (5.7).

Lemma 6. Let μ be a finite Borel measure on $[0,1]$. Let ${}^*\mathbb{R}$ be a non-standard extension of $\mathbb{R}$, and let $P < N$ be a hyperfinite positive integer such that $P/N \sim 0$.

Then there exists a vector $q \in {}^*\mathbb{R}^{N+1}$ such that for every continuous function Φ on $[0,1]$,

$$(5.8) \qquad \int_0^1 \Phi d\mu = st(N^{-1}\Sigma\, {}^*\Phi(k/N)q(k))$$

and that

$$(5.9) \qquad 0 \leq q(k) \leq P.$$

Proof. Let M be a finite integer such that $\mu[0,1] < M$.

Let $Q(x) = \mu[0,x]$ be the cumulative distribution function of μ, and let

$$\tilde{q}(k) = {}^*Q(k/N).$$

Let P_1 denote the hyperfinite integer which satisfies $N \leq P_1 P < N+P$, and note that $MP_1 < N-MP_1$ when M is finite. Define

$$q(k) = \begin{cases} \frac{N}{MP_1+1}\, \tilde{q}(k+MP_1) & , \ 0 \leq k < MP_1 \\ \frac{N}{MP_1+1}\, (\tilde{q}(k+MP_1) - \tilde{q}(k)), & MP \leq k \leq N - MP_1 \\ \frac{N}{MP_1+1}\, (\tilde{q}(N) - \tilde{q}(k)) & , \ N - MP_1 < k \leq N. \end{cases}$$

Then clearly (5.9) is true, and the verification of (5.8) is straight-forward (use that F is uniformly continuous).

Now we return to the boundary value problem (2.1), (2.2) of Section 2.

We choose a vector $q \in {}^*\mathbb{R}^{N+1}$ to represent the measure μ from Section 2, as described in Lemma 6, with $P^4 < N$, and consider the discrete boundary value problem (3.1), (3.2) on the interval $\{0,1,\ldots,N\}$ in ${}^*\mathbb{Z}$. (q and y now take their values in ${}^*\mathbb{R}$).

The "transfer principle" or "elementary extension principle" of non-standard analysis (see [5]) then tells us that all the results we found in Section 3 and 4 about the eigenvalues and eigenfunctions of (3.1), (3.2) remain valid in our present, non-standard setting. We will prove Theorem 1 by translating them back to the standard setting of Section 2.

First, from (3.28) it follows, since $m = \|q\|_1/N < \infty$, that λ_j is finite if and only if j is finite, and since in that case

$\mathrm{st}(\sigma_j) = \pi^2 j^2$ by (3.19), we obtain

$$\pi^2 j^2 \leq \tau_j = \mathrm{st}(\lambda_j) \leq (\pi j + M)^2 .$$

That is, the numbers $\tau_j = \mathrm{st}(\lambda_j)$ satisfy a) of Theorem 1.

Next, use Proposition 3: From (4.3) it follows that

$$|y_j(m) - y_j(1)| = \left|\sum_{k=1}^{m-1} \Delta y_j(k)\right| \leq (m-1)\lambda_j^{\frac{1}{2}} B/N.$$

This implies that when j is finite, (5.2) is valid for y_j, and hence that the functions

$$(5.10) \qquad Y_j(x) = \mathrm{st}(y_j(k)) \quad \text{when} \quad x = \mathrm{st}(k/N)$$

are well defined and continuous on $[0,1]$ (in fact they are even Lipschitz-continuous). The bound (4.1) then implies

$$|Y_j(x)| \leq 2(1 + 12M)$$

for all x, all finite j, and a) is proved.

In view of (5.4), the orthogonality relations (3.5), (3.6) now imply that the functions $\{Y_j\}_{j=1}^{\infty}$ are orthonormal over $[0,1]$ (with respect to Lebesgue measure) as stated in b) of Theorem 1.

To prove completeness, we consider a two times continuously differentiable function Φ on $[0,1]$, with $\Phi(0) = \Phi(1) = 0$, and we define the corresponding $\varphi \in {}^*\mathbb{R}^{N+1}$ by (5.1).

Then, from Proposition 1 we have

$$\varphi(k) = N^{-1}\sum_{j=1}^{N-1} c_j y_j(k)$$

with

$$c_j = \langle \varphi, y_j \rangle = \sum_{k=o}^{N} \varphi(k) y_j(k) .$$

For finite j, we have

$$(5.11) \quad d_j = st(c_{j/N}) = st(N^{-1} \Sigma \varphi(k)y_j(k)) = \int_0^1 \Phi\, Y_j dm.$$

It follows that for any positive integer $M < \infty$

$$st(N^{-1} \sum_{j=1}^{M} c_j y_j(k)) = \sum_{j=1}^{M} d_j Y_j(st(k/N))$$

with d_j defined by (5.11).

To prove completeness for $\{Y_j\}_{j=1}^{\infty}$, it will therefore be sufficient to show that for every real positive standard ε there is an integer $M = M_\varepsilon < \infty$ such that

$$(5.12) \qquad N^{-1} \sum_{j=M}^{N-1} |c_j\, y_j(k)| < \varepsilon\ , \qquad 0 \leq k \leq N.$$

To do that, we need a good inequality for c_j. From (3.8) and (3.1) we find

$$(5.13) \quad \begin{aligned} c_j &= \langle \varphi, y_j \rangle = \langle \varphi, \lambda_j^{-1}(y_j q - N^2 \Delta^2 y_j)\rangle \\ &= \lambda_j^{-1}[\sum_{k=0}^{N} \varphi(k) y_j(k) q(k) - N^2 \sum_0^N \varphi(k)\, \Delta^2 y_j(k)] \end{aligned}$$

The assumption that Φ and Φ'' are continuous on $[0,1]$, implies that for some (finite) real p,

$$|\varphi(k)| \leq p \quad \text{and} \quad N^2 |\Delta^2 \varphi(k)| \leq p\ , \qquad 0 \leq k \leq N.$$

For the two terms in the square bracket in (5.13) this implies

$$|\sum_{k=0}^{N} \varphi(k) y_j(k) q(k)| \leq p \|q\|_1 \|y_j\|_\infty$$

and, using summation by parts two times:

$$|\Sigma\, \varphi(k) \Delta^2 y_j(k)| = |\Sigma\, y_j(k) \Delta^2 \varphi(k)| \leq p \|y_j\|_\infty / N.$$

From (5.13)

$$|c_j| \leq \lambda_j^{-1} p \|y_j\|_\infty (\|q\|_1 + N)$$

and hence

$$N^{-1} \sum_{j=M}^{N-1} |c_j y_j(k)| \leq p(1 + \|q\|_1/N) \sum_{M}^{N-1} \|y_j\|_\infty^2 \lambda_j^{-1} .$$

Now, from Proposition 3:

$$\sum_{M}^{N-1} \frac{\|y_j\|_\infty^2}{\lambda_j} \leq \sum_{M}^{N/2} \frac{b}{\pi^2 j^2} + \sum_{n/2}^{N-1} \frac{b'}{\pi^2 j^c}$$

(5.14)

$$< \frac{b}{\pi^2} \sum_{j=M}^{N/2} j^{-2} + \frac{b'}{\pi^2} \cdot \frac{N}{2} \cdot \left(\frac{2}{N}\right)^2$$

Recall from Proposition 3 that b is finite, and that $b' \leq 24(\|q\|_\infty + 1)$. Since Lemma 6 implies that we may suppose $\|q\|_\infty < N^{1/4}$, the last term in (5.14) is infinitesimal, and hence (5.12) will be true when $M < \infty$ is large enough. This proves c) in Theorem 1, and the completeness statement in b) as well.

To prove d), let Φ be a continuously differentiable function on $[0,1]$, with $\Phi(0) = \Phi(1) = 0$, and define $\varphi \in {}^*\mathbb{R}^{N+1}$ by (6.1) as before.

Expand φ in terms of $\{y_j\}$, take second differences, and use (3.1):

$$N^2 \Delta^2 \varphi(k) = N^2 \sum_{j=1}^{N-1} c_j \Delta^2 y_j(k) =$$

$$= \sum_{j=1}^{N} c_j y_j(k)(q(k) - \lambda_j) .$$

Next multiply by $\varphi(k)$, take the sum over $k: 0 \leq k \leq N$, and use summation by parts in the lefthand side. This gives:

$$-N^2\sum_{k=o}^{N-1}\Delta\varphi(k)^2 = \sum_{j=1}^{N-1} c_j \sum_{k=1}^{N-1} y_j(k)\varphi(k)q(k) - \sum_j \lambda_j c_j \sum_k y_j(k)\varphi_j(k)$$

$$= \sum_k \varphi\ (k)^2 q(k) - \sum_j \lambda_j c_j^2 .$$

Finally, divide by N and take standard parts:

$$\int(\Phi'(x))^2 dx + \int \Phi^2\, d\mu = \Sigma\, \tau_j\, d_j^2$$

and d) is proved.

To prove e), take the sum in (3.1) over all k, $0 \leq k \leq m$; and obtain

$$\Delta y_j(m) = \Delta y_j(0) + N^{-2} \sum_{k=1}^{n} (q(k) - \lambda) y_j(k) .$$

Then take the sum over m, $0 \leq m < n$ and interchange the order of summation:

$$y_j(n) = n\Delta y_j(0) + N^{-2} \sum_{k=1}^{n} (n-k) y_j(k) q(k) - \lambda N^{-2} \sum_{k=1}^{n} (n-k) y_j(k).$$

Let j be finite, and take standard parts:

$$(5.14)\quad Y_j(x) = x\varkappa + \int_0^x (x-t) Y_j(x)\, d\mu(x) - \tau_j \int_0^x (x-t) Y_j(x)\, dm(x)$$

with $x = st(n/N)$, $t = st(k/N)$, $\tau_j = st(\lambda_j)$ and $\varkappa = st(N\Delta y(0))$.

This proves e) in Theorem 1.

To prove the oscillation and separation theorem, i.e. item f) of Theorem 1, we recall from Proposition 2 that y_j has exactly $k+1$ zero-points

$$0 = \xi_{j,o} < \xi_{j,1} < \ldots < \xi_{j,j} = N .$$

Each of these of course gives a zero

$$x_{j,k} = st(\xi_{j,k}/N)$$

for Y_j, and from Lemma 6 it follows that they are distinct:

$$x_{j,k} - x_{j,k-1} \geq \pi\ st(6\lambda_j)^{-\frac{1}{2}} > 0.$$

In addition, it follows from Lemma 7 that

$$x_{j+1,k} \leq x_{j,k} \leq x_{j+1,k+1} .$$

It remains to show that Y_j cannot have any additional zeroes, and that Y_j and Y_{j+1} cannot have a common zero in $\langle 0,1 \rangle$. Both these facts are best proved by standard methods, starting from (2.4), and since the proofs follow [2] quite closely, it seems unnecessary to reproduce them here.

Finally, if μ is absolutely continuous with respect to Lebesgue measure on some interval $I \subset [0,1]$, $\mu = g dm$ on I, with g integrable on I, then the integral equation (2.4) shows that the Y_j are continuously differentiable there:

$$Y_j'(x) = \varkappa + \int_{x_0}^{x} Y_j(x)(g(x) - \tau)dm(x)$$

and since g is integrable, Y_j' is absolutely continuous, hence differentiable almost everywhere, and Y_j solves (2.1) in the ordinary sense almost everywhere on I. The inequality for Y_j' follows from (4.3).

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